Title: Generating random variates from truncated distributions

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Short Description:

For some simulations random variates have to be generated from a truncated distribution with given mean. This can be done utilizing the non-truncated distribution, but often only the mean of the non-truncated random variables is adjustable such that it has to be modified properly.

Keywords: simulation, random variates, truncated distribution

Example:

Consider the nonnegative random variable $\tilde{X}$ with mean value $E(\tilde{X})$, which is described through its distribution function $F_{\tilde{X}}(x) = \int_{x'=0}^{x} \tilde{f}_{\tilde{X}}(x')dx'$ and density $f_{\tilde{X}}(x)$ for $0 \le x \le \infty$. In order to generate a truncated distribution $F_X(x)$, $a \le x \le b$, with mean $E(X)$ the following methods are commonly used.

Method1: Scaled truncation

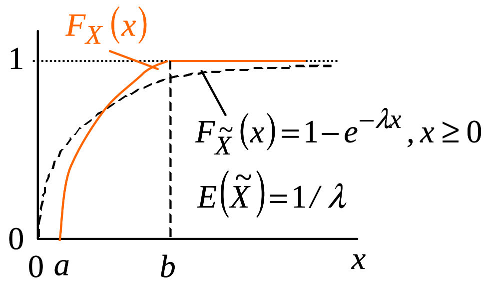
If the inverse transform method can be applied to $F_{\tilde{X}}(x)$ an algorithm for generating X is as follows [1]:

Generate $U \sim U[0,1]$.
Let $V = F_{\tilde{X}}(a) + \left[F_{\tilde{X}}(b) - F_{\tilde{X}}(a)\right] \cdot U$.
Return $X = F_{\tilde{X}}^{-1}(V)$.

The truncated distribution function is given by

$$F_X(x) = \begin{cases} 0 & \text{for} \quad x < a \\ \dfrac{F_{\tilde{X}}(x) - F_{\tilde{X}}(a)}{F_{\tilde{X}}(b) - F_{\tilde{X}}(a)} & \text{for} \quad a \le x \le b \\ 1 & \text{for} \quad x > b \end{cases}$$

Figure 1 gives an example for the exponential distribution.



$$E(X) = E(\tilde{X}) + a - \frac{(b-a)\left(1 - F_{\tilde{X}}(b)\right)}{F_{\tilde{X}}(b) - F_{\tilde{X}}(a)}$$

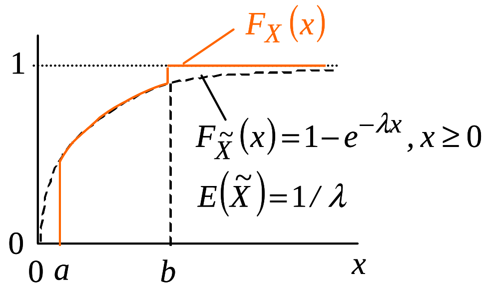
Figure 1: Exponential distribution $\tilde{F}_X(x)$ and the scaled truncated distribution $F_X(x)$ with mean value $E(X)$.

Method2: Cut-off truncation

Assume we have a method for generating variates from $F_{\tilde{X}}(x)$. The truncation is done as follows:

Generate X from $F_{\tilde{X}}(x)$.
If $X \le a$ let $X = a$.
If $X \ge b$ let $X = b$.

In this case the truncated distribution function is given by [2]



$$E(X) = a + E(\tilde{X}) \cdot \left(F_{\tilde{X}}(b) - F_{\tilde{X}}(a)\right)$$

Figure 2: Exponential distribution $\tilde{F}_X(x)$ and the cut-off truncated distribution $F_X(x)$ with mean value $E(X)$.

$$F_X(x) = \begin{cases} 0 & \text{für} \quad x < a \\ F_{\tilde{X}}(x) & \text{für} \quad a \le x < b \\ 1 & \text{für} \quad x \ge b \end{cases}$$

Figure 2 gives an example for the truncated exponential distribution.

References:

[1] G. S. Fishman; 'Concepts And Methods In Diskrete Event Digital Simulation'; Wiley & Sons, 1973.

[2] A. M. Law; W. D. Kelton; 'Simulation Modelling & Analysis', McGraw Hill Inc, 1991.